Lefshcetz-thimble path integral for studying the Silver Blaze phenomenon

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Introduction and Motivation

Where is the finite-density QCD?

Neutron star

- Cold and dense nuclear matters
- $2m_{\mathrm{sun}}$ neutron star (2010)
- Gravitational-wave observations (2017$\sim$)

Heavy-ion collision

- Low-energy scan of heavy-ion collisions are planned and run in many facilities (RHIC, SIS, J-PARC)

Reliable theoretical approach to equation of state must be developed!

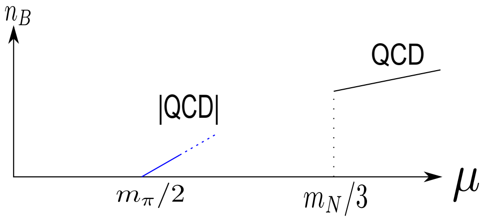
$$Z(T,\mu)=\int \mathcal{D}A\,\mathrm{Det}(\not{D}(A)+m)\exp -S_{\mathrm{YM}}(A).$$

Silver Blaze problem in finite-density QCD

QCD & |QCD|

$$Z_{\mathrm{QCD}}=\int\mathcal{D}A\,(\mathrm{det}\gamma_\nu D_\nu)\,\mathrm{e}^{-S_{\mathrm{YM}}},\ Z_{|\mathrm{QCD}|}=\int\mathcal{D}A\,|\mathrm{det}\gamma_\nu D_\nu|\,\mathrm{e}^{-S_{\mathrm{YM}}}.$$

At $\mu=0$, these two are the same! But,



At $T=0$, the state must be equal to the QCD vac. for $|\mu|\lesssim m_N/3$.
Can we describe this "trivial" behavior using path integral?

Lefschetz decomposition formula

Oscillatory integrals with **many variables** can be evaluated using the "steepest descent" cycles $\mathcal{J}_\sigma$: (classical eom $S'(z_\sigma)=0$)

$$\int_{\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{e}^{-S(x)} = \sum_\sigma \langle \mathcal{K}_\sigma, \mathbb{R}\rangle \int_{\mathcal{J}_\sigma} \mathrm{d}^n z\, \mathrm{e}^{-S(z)}.$$

$\mathcal{J}_\sigma$ are called Lefschetz thimbles, and $\mathrm{Im}[S]$ is constant on it:

$$\mathcal{J}_\sigma = \left\{ z(0) \middle| \lim_{t\to-\infty} z(t) = z_\sigma \right\}, \quad \frac{\mathrm{d}z^i(t)}{\mathrm{d}t} = \overline{\left(\frac{\partial S(z)}{\partial z^i}\right)}.$$

$\langle \mathcal{K}_\sigma, \mathbb{R}\rangle$: intersection numbers of duals $\mathcal{K}_\sigma$ and $\mathbb{R}^n$
($\mathcal{K}_\sigma = \{z(0) | z(\infty) = z_\sigma\}$).

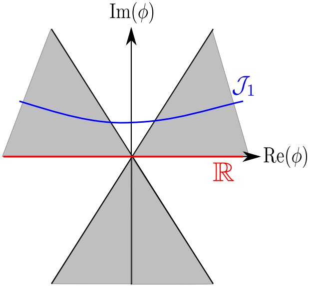
[Witten, arXiv:1001.2933, 1009.6032]

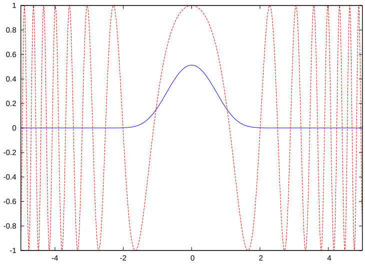
Lefschetz thimble for Airy integral

Airy integral is given as

$$\mathrm{Ai}(a)=\int_{\mathbb{R}}\frac{\mathrm{d}x}{2\pi}\exp\mathrm{i}\left(\frac{x^3}{3}+ax\right)$$

Complexify the integration variable: $z=x+\mathrm{i}y$.





Integrand on $\mathbb{R}$, and on $\mathcal{J}_1$ $(a=1)$

Rewrite the Airy integral

There exists two Lefschetz thimbles $\mathcal{J}_\sigma$ $(\sigma = 1, 2)$ for the Airy integral:

$$\mathrm{Ai}(a) = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \frac{\mathrm{d}z}{2\pi} \exp \mathrm{i} \left(\frac{z^3}{3} + az \right).$$

n_σ: intersection number of the steepest ascent contour $\mathcal{K}_\sigma$ and $\mathbb{R}$.

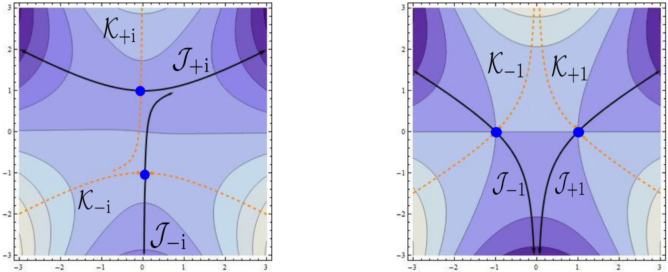


Figure: Lefschetz thimbles $\mathcal{J}$ and duals $\mathcal{K}$ $(a = 1\mathrm{e}^{0.1\mathrm{i}}, -1)$

Simplest model study to understand the sign problem

One-site Fermi Hubbard model

One-site Hubbard model:

$$\hat{H} = U\hat{n}_\uparrow \hat{n}_\downarrow - \mu(\hat{n}_\uparrow + \hat{n}_\downarrow).$$

Fock state gives the number density immediately:

$$\langle \hat{n} \rangle = \frac{1}{\beta}\frac{\partial}{\partial \mu}\ln Z = \frac{2(\mathrm{e}^{\beta\mu} + \mathrm{e}^{\beta(2\mu-U)})}{1 + 2\mathrm{e}^{\beta\mu} + \mathrm{e}^{\beta(2\mu-U)}}.$$

In the zero-temperature limit,

$$n(\beta=\infty) = \begin{cases} 2 & (1 < \mu/U), \\ 1 & (0 < \mu/U < 1), \\ 0 & (\mu/U < 0). \end{cases}$$

Path integral for one-site model

The path-integral expression for the one-site Hubbard model: :

$$Z=\sqrt{\frac{\beta}{2\pi U}}\int_{\mathbb{R}}\mathrm{d}\varphi\,\underbrace{\left(1+\mathrm{e}^{\beta(\mathrm{i}\varphi+\mu+U/2)}\right)^2}_{\text{Fermion Det}}\mathrm{e}^{-\beta\varphi^2/2U}.$$

Integrand has complex phases causing the sign problem.

φ is an auxiliary field for the number density:

$$\langle\hat{n}\rangle=\mathrm{Im}\langle\varphi\rangle/U.$$

Flows at $\mu/U<-0.5$ and $\mu/U>1/5$

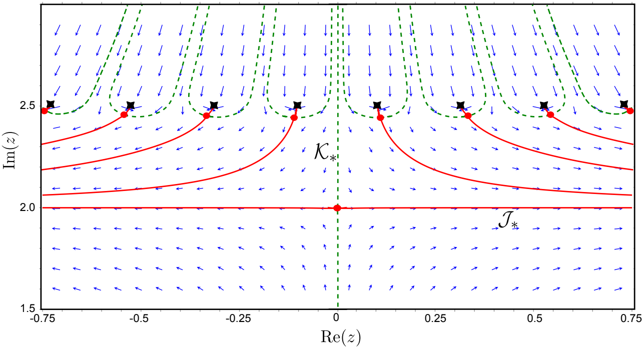


Figure: Flow at $\mu/U=2$

$$Z=\int_{\mathcal{J}_*}\mathrm{d}z\ \mathrm{e}^{-S(z)}.$$

Number density: $n_*=0$ for $\mu/U<-0.5$, $n_*=2$ for $\mu/U>1.5$.

(YT, Hidaka, Hayata, 1509.07146)

Flows at $-0.5 < \mu/U < 1.5$

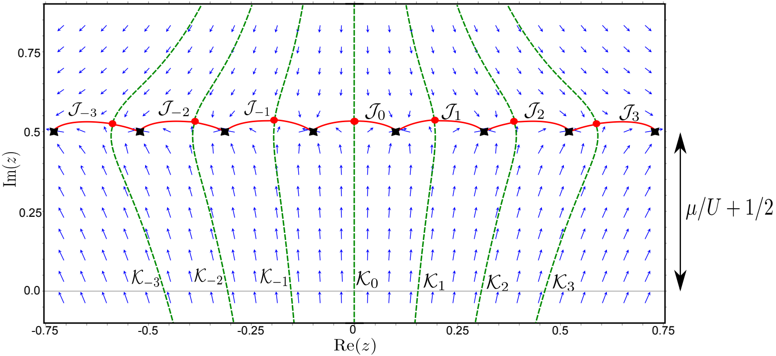


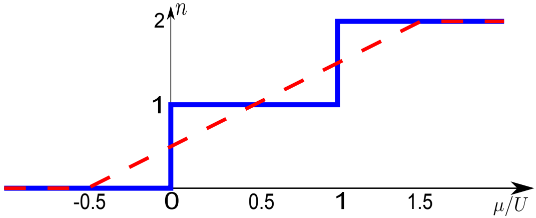
Figure: Flow at $\mu/U = 0$

Complex saddle points lie on $\mathrm{Im}(z_m)/U \simeq \mu/U + 1/2$.

This value is far away from $n = \mathrm{Im}\,\langle z \rangle/U$=0, 1, or 2.

Curious incident of n in the one-site model

We have a big difference between the exact result and naive expectation:



(YT, Hidaka, Hayata, arXiv:1509.07146[hep-th])
(cf. Monte Carlo with one-thimble approx. gives the naively expected results.

Fujii, Kamata, Kikukawa,1509.08176, 1509.09141; Alexandru, Basar, Bedaque,1510.03258)

Question

Can we unveil this curious incident?

Semiclassical study of the curious incident

Complex classical solutions

Let us solve, for $-0.5 < \mu/U < 1.5$,

$$\frac{\beta}{U}\mathrm{i}z + \frac{2\beta \exp\beta\left(\mathrm{i}z + \mu + \frac{U}{2}\right)}{1 + \exp\beta\left(\mathrm{i}z + \mu + \frac{U}{2}\right)} = 0.$$

If $\beta U \gg 1$, the solutions are labeled by $m \in \mathbb{Z}$:

$$z_m \simeq \mathrm{i}\left(\mu + \frac{U}{2}\right) + 2\pi m T.$$

At these solutions, the classical actions become

$$S_0 \simeq -\frac{\beta U}{2}\left(\frac{\mu}{U} + \frac{1}{2}\right)^2,$$
$$\mathrm{Re}\left(S_m - S_0\right) \simeq \frac{2\pi^2}{\beta U} m^2,$$
$$\mathrm{Im}\, S_m \simeq 2\pi m \left(\frac{\mu}{U} + \frac{1}{2}\right).$$

Semiclassical partition function

Using complex classical solutions z_m, let us calculate

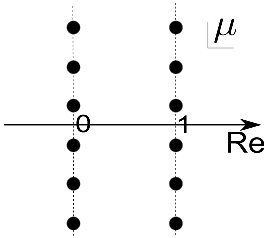
$$Z_{\mathrm{cl}} := \sum_{m=-\infty}^{\infty} \mathrm{e}^{-S_m}.$$

This expression is valid for $-1/2 \lesssim \mu/U \lesssim 3/2$.
This is calculable using the elliptic theta function:

$$\begin{aligned} Z_{\mathrm{cl}} &\simeq \mathrm{e}^{-S_0}\left(1+2\sum_{m=1}^{\infty}\cos 2\pi m\left(\frac{\mu}{U}+\frac{1}{2}\right)\mathrm{e}^{-2\pi^2 m^2/\beta U}\right) \\ &= \mathrm{e}^{-S_0}\theta_3\left(\pi\left(\frac{\mu}{U}+\frac{1}{2}\right), \mathrm{e}^{-2\pi^2/\beta U}\right). \end{aligned}$$

Number density & Lee–Yang zeros

Lee–Yang zeros of $Z_{\rm cl}$:



Semiclassical study gives the correct transition!

$$n_{\rm cl}:={1\over \beta}{\partial\over \partial\mu}\ln Z_{\rm cl}\to \begin{cases}2 & (1<\mu/U<3/2),\\ 1 & (0<\mu/U<1),\\ 0 & (-1/2<\mu/U<0).\end{cases}$$

(YT, Hidaka, Hayata, arXiv:1509.07146[hep-th])

Important interference among multiple thimbles

Let us consider a "phase-quenched" multi-thimble approximation:

$$Z_{|\mathrm{cl.}|}=\sum_m \left|\mathrm{e}^{-S_m}\right| = \mathrm{e}^{-S_0(\mu)}\theta_3(0,\mathrm{e}^{-2\pi^2/\beta U}).$$

- Lee–Yang zeros cannot appear at $\mu/U=0,\,1$.
- One-thimble, or "phase-quenched", result: $n\simeq \mu/U+1/2$.

Consequence

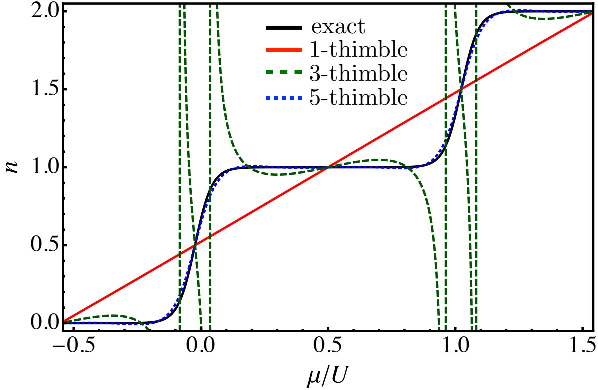
In order to describe the step functions, we need interference of complex phases among different Lefschetz thimbles.

(cf. Particle Productions: Dumulu, Dunne, PRL 104 250402)

(cf. Hidden Topological Angles: Behtash, Sulejmanpasic, Schäfer, Ünsal, PRL 115 041601)

Numerical results

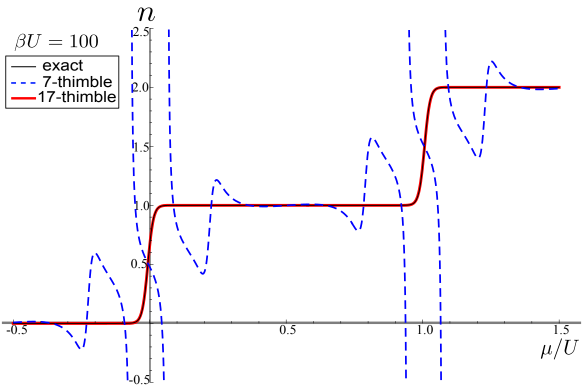
Results for $\beta U = 30$: (1, 3, 5-thimble approx.: $\mathcal{J}_0$, $\mathcal{J}_0 \cup \mathcal{J}_{\pm 1}$, and $\mathcal{J}_0 \cup \mathcal{J}_{\pm 1} \cup \mathcal{J}_{\pm 2}$)



(YT, Hidaka, Hayata, arXiv:1509.07146[hep-th])

Numerical results

Results for $\beta U = 100$:



Necessary numbers of Lefschetz thimbles $\propto \beta U/2\pi$.

(YT, Hidaka, Hayata, arXiv:1509.07146[hep-th])

Summary and conclusion

Summary and Conclusion

- Picard–Lefschetz theory gives a suitable framework for saddle-point analysis even if $S(\phi)$ takes complex values.
- One-site Hubbard model is a nice toy model to play with the sign problem.
- Destructive interference among complex phases of Lefschetz thimbles play a pivotal role for the baryon Silver Blaze.